Examples:

① $x[n] = \delta[n]$

$$X(z) = \sum_{n=0}^{\infty} \delta[n] z^{-n} = \delta[0] z^{-0} = 1 \quad \Rightarrow \boxed{\delta[n] \longleftrightarrow 1}$$

②

$x[n] = \delta[n-q]$, where q is positive integer

$$X(z) = \sum_{n=0}^{\infty} \delta[n-q] z^{-n} = \delta[0] z^{-q} = \frac{1}{z^q} \quad \Rightarrow \boxed{\delta[n-q] \longleftrightarrow \frac{1}{z^q}}$$

③

$x[n] = u[n]$

$$X(z) = \sum_{n=0}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = 1 + z^{-1} + z^{-2} + \cdots$$

$$\therefore (z-1) X(z) = (z-1)(1 + z^{-1} + z^{-2} + \cdots)$$

$$= (z + 1 + z^{-1} + z^{-2} \cdots) - (1 + z^{-1} + z^{-2} + \cdots)$$

$$= z$$

$$\Rightarrow X(z) = \frac{z}{z-1} \quad \Rightarrow \boxed{u[n] \longleftrightarrow \frac{z}{z-1}}$$

④

$x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = 1 + a z^{-1} + a^2 z^{-2} + \cdots$$

$$\therefore (z-a) X(z) = (z-a)(1 + a z^{-1} + a^2 z^{-2} + \cdots)$$

$$= (z + a + a^2 z^{-1} + \cdots) - (a + a^2 z^{-1} + a^3 z^{-2} + \cdots)$$

$$= z$$

$$\therefore X(z) = \frac{z}{z-a} \quad \Rightarrow \boxed{a^n u[n] = \frac{z}{z-a}}$$

⑤ $x[n] = u[n] + a^n u[n]$

$$X(z) = \frac{z}{z-1} + \frac{z}{z-a} = \frac{2z^2 - (1+a)z}{(z-1)(z-a)}$$

⑥ $x[n] = u[n] - u[n-2]$

$$X(z) = \frac{z}{z-1} - \frac{z}{z-1} z^{-2} = \frac{z - z^{-1}}{z-1}$$

⑦ $x[n] = u[n+1]$

$$X(z) = z X(z) - x[0] z$$

$$= z\left(\frac{z}{z-1}\right) - z = \frac{z^2 - z^2 + z}{z-1} = \frac{z}{z-1}$$

⑧ $x[n] = n a^n u[n]$

Let $v[n] = a^n u[n] \Rightarrow V(z) = \frac{z}{z-a}$

$$\therefore X(z) = -z \frac{d}{dz} V(z) = -z \frac{d}{dz}\left(\frac{z}{z-a}\right)$$

$$= -z \frac{(z-a)(1) - (z)(1)}{(z-a)^2} = -\frac{z(-a)}{(z-a)^2}$$

$$\therefore X(z) = \frac{az}{(z-a)^2}$$

If $a=1 \Rightarrow X(z) = \frac{z}{(z-a)^2} \leftrightarrow n u[n]$

$$\Rightarrow \boxed{n a^n u[n] \leftrightarrow \frac{az}{(z-a)^2}} \neq \boxed{n u[n] \leftrightarrow \frac{z}{(z-a)^2}}$$

(9)

$x[n] = n^2 a^n u[n]$

let $v[n] = n a^n u[n] \Rightarrow V(z) = \dfrac{az}{(z-a)^2}$ (from Example 8)

$\therefore X(z) = -z \dfrac{d}{dz} V(z) = -z \dfrac{d}{dz}\left(\dfrac{az}{(z-a)^2}\right)$

$= \dfrac{-z\left[(z-a)^2(a) - (az)(2(z-a))\right]}{(z-a)^4}$

$= \dfrac{-z(z-a)[az - a^2 - 2az]}{(z-a)^4} = -z\,\dfrac{-az-a^2}{(z-a)^3}$

$= \dfrac{az^2 + a^2 z}{(z-a)^3} = \dfrac{az(z+a)}{(z-a)^3}$

If $a=1$, $X(z) = \dfrac{z(z+1)}{(z-1)^3}$

$\therefore$ $n^2 a^n u[n] \leftrightarrow \dfrac{az(z+a)}{(z-a)^3}$ $\neq$ $n^2 u[n] \leftrightarrow \dfrac{z(z+1)}{(z-1)^3}$

(10) $x[n] = a^n p[n]$ where $p[n] = u[n] - u[n-q]$

$x[n] = a^n (u[n] - u[n-q])$

$u[n] - u[n-q] \leftrightarrow \dfrac{z(1-z^{-q})}{z-1}$

$\therefore X(z) = \dfrac{(z/a)(1-(z/a)^{-q})}{z/a - 1} = \dfrac{z(1-a^q z^{-q})}{z-a}$

or $a^n u[n] \Leftrightarrow \frac{z}{z-a}$

$$u[n-q] \Leftrightarrow \frac{z}{z-1} z^{-q} = \frac{z^{-q+1}}{z-1}$$

$$a^n u[n-q] \Leftrightarrow \frac{(z/a)^{-q+1}}{(z/a)-1} = \frac{z^{-q+1}\, a^{q-1}}{z/a-1} = \frac{a^q z^{-q+1}}{z-a}$$

$$\Leftrightarrow \frac{z a^q z^{-q}}{z-a}$$

$$\therefore X(z) = \frac{z}{z-a} - \frac{z a^q z^{-q}}{z-a} = \frac{z(1-a^q z^{-q})}{z-a} \quad \#$$

III $x[n] = \cos(\Omega n) u[n]$

Let $v[n] = u[n] \Rightarrow V(z) = \frac{z}{z-1}$

$$\therefore X(z) = \frac{1}{2}\left(\frac{e^{j\Omega} z}{e^{j\Omega} z - 1} + \frac{e^{-j\Omega} z}{e^{-j\Omega} z - 1} \right)$$

$$= \frac{1}{2}\left[\frac{e^{j\Omega} z (e^{-j\Omega} z - 1) + e^{-j\Omega} z (e^{j\Omega} z - 1)}{(e^{j\Omega} z - 1)(e^{-j\Omega} z - 1)} \right]$$

$$= \frac{1}{2}\left[\frac{z^2 - e^{j\Omega} z + z^2 - e^{-j\Omega} z}{z^2 - z(e^{j\Omega} + e^{-j\Omega}) + 1} \right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - 2\cos(\Omega) z}{z^2 - 2\cos(-\Omega) z + 1} \right] = \frac{z^2 - \cos(\Omega) z}{z^2 - 2\cos(\Omega) + 1}$$

12 $x[n] = a^n \cos(\Omega n)\, u[n]$

Let $U[n] = \cos(\Omega n)\, u[n]$

$$\Rightarrow U(z) = \frac{z^2 - \cos(\Omega) z}{z^2 - 2\cos(\Omega) z + 1}$$

$$\therefore X(z) = \frac{(z/a)^2 - \cos(\Omega)(z/a)}{(z/a)^2 - 2\cos(\Omega)(z/a) + 1}$$

$$= \frac{z^2 - a\cos(\Omega) z}{z^2 - 2a\cos(\Omega) z + a^2}$$

13 $$X(z) = \frac{z}{8z^2 - 2z + 1}$$

$V[n] = e^{3n} x[n]$

$V(z) = ?$

$$V(z) = X(e^{-3} z) = \frac{e^{-3} z}{8e^{-6} z^2 - 2e^{-3} z + 1}$$

14 $x[n] = e^{0.5n} u[n] + u[n-2]$

$$X(z) = \frac{z}{z - e^{0.5}} + \frac{z}{z-1} z^{-2}$$

(1) A system is described by the difference equation

$$y[n] + 0.5\, y[n-1] = u[n]; \quad y[-1] = 0$$

Find an analytical expression for $y[n]$ and the transfer function of the system

$$Y(z) + 0.5\, z^{-1} Y(z) = \frac{z}{z-1}$$

$$Y(z) = \left(\frac{z}{z-1}\right)\left(\frac{1}{1+0.5z^{-1}}\right) = \underbrace{\left(\frac{z}{z-1}\right)\left(\frac{z}{z+0.5}\right)}_{H(z)}$$

$$= \frac{z^2}{(z-1)(z+0.5)}$$

$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z+0.5)} = \frac{C_1}{z-1} + \frac{C_2}{z+0.5}$$

$$C_1 = (z-1)\left.\frac{Y(z)}{z}\right|_{z=1} = \left.\frac{z}{z+0.5}\right|_{z=1} = \frac{2}{3}$$

$$C_2 = (z+0.5)\left.\frac{Y(z)}{z}\right|_{z=-0.5} = \left.\frac{z}{z-1}\right|_{z=-0.5} = \frac{1}{3}$$

$$\therefore Y(z) = \frac{2}{3}\left(\frac{z}{z-1}\right) + \frac{1}{3}\left(\frac{z}{z+0.5}\right)$$

$$\therefore y[n] = \frac{2}{3}u[n] + \frac{1}{3}(-0.5)^n u[n]$$

or $$y[n] = \frac{2}{3} + \frac{1}{3}(-0.5)^n, \quad n \geq 0$$

$$\boxed{H(z) = \frac{z}{z+0.5}}$$

(2) Find the transfer function of the following system if all ICs = 0

$$y[n] - 0.2\,y[n-1] - 0.8\,y[n-2] = 2x[n] + x[n-1]$$

$$Y(z) - 0.2\,z^{-1}Y(z) - 0.8\,z^{-2}Y(z) = 2X(z) + z^{-1}X(z)$$

$$\therefore Y(z)\left[1 - 0.2\,z^{-1} - 0.8\,z^{-2}\right] = X(z)\left[2 + z^{-1}\right]$$

$$\therefore \frac{Y(z)}{X(z)} = H(z) = \frac{2 + z^{-1}}{1 - 0.2\,z^{-1} - 0.8\,z^{-2}} \times \frac{z^2}{z^2}$$

$$\therefore H(z) = \frac{2z^2 + z}{z^2 - 0.2\,z - 0.8}$$

(3) A LTI DT system is given by the input/output difference equation

$$y[n] + y[n-1] - 2\,y[n-2] = 2x[n] - x[n-1] \quad -(1)$$

Find $Y(z)$ if $y[-2] = 2$ and $y[-1] = 0$

- From (1):

$$Y(z) + z^{-1}Y(z) + y[-1] - 2\left(z^{-2}Y(z) + y[-2] + z^{-1}y[-1]\right)$$
$$= 2X(z) - z^{-1}X(z) - x[-1]$$

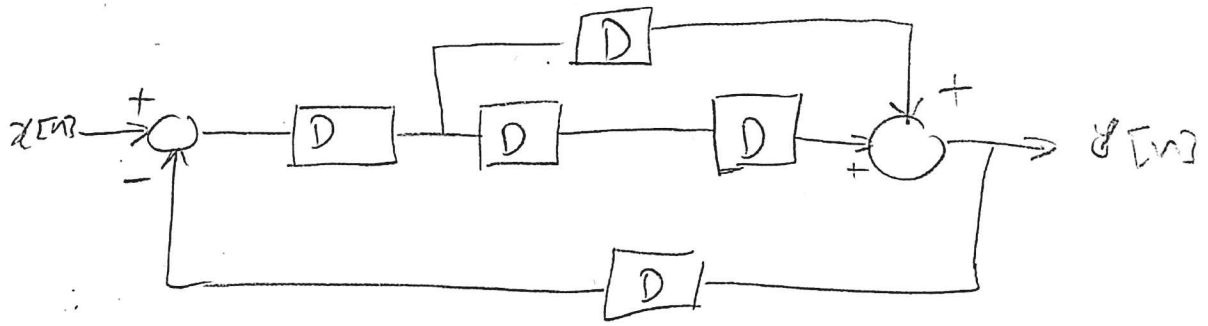
$$\therefore Y(z) + z^{-1}Y(z) - 2z^{-2}Y(z) - 4 = 2X(z) - z^{-1}X(z)$$

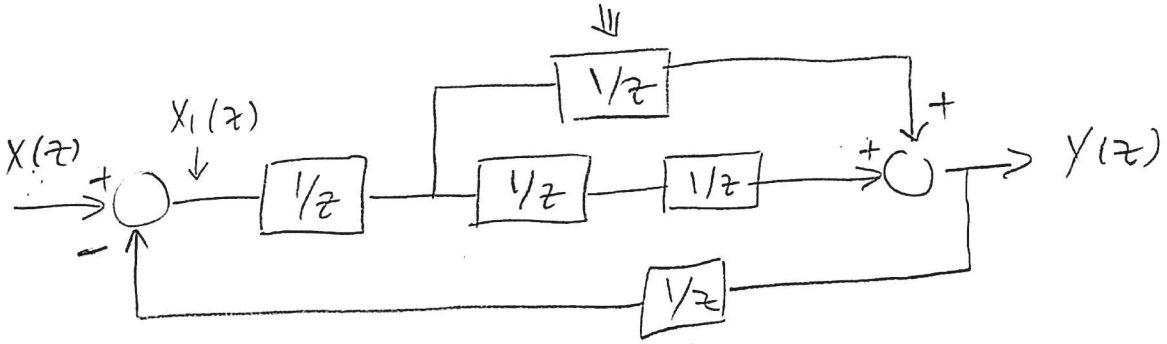
$$Y(z)\left(1 + z^{-1} - 2z^{-2}\right) - 4 = X(z)\left(2 - z^{-1}\right)$$

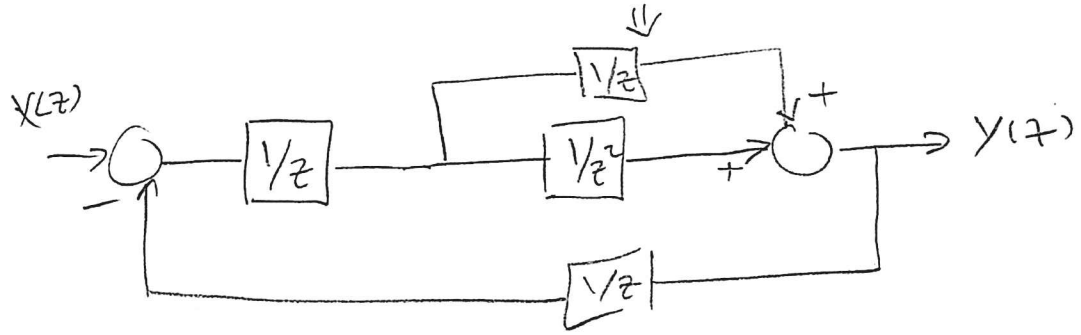
$$\therefore Y(z) = \frac{4}{1 + z^{-1} - 2z^{-2}} + \frac{2 - z^{-1}}{1 + z^{-1} - 2z^{-2}}X(z)$$

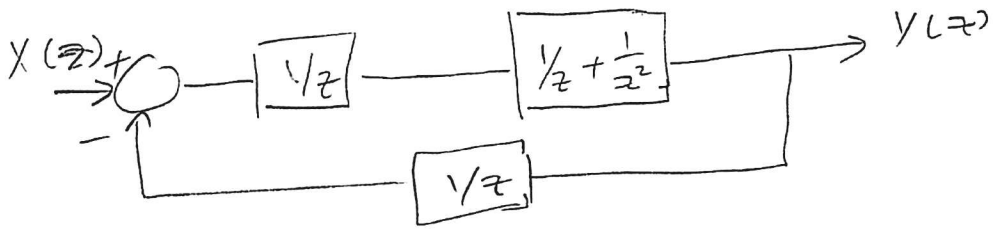
$$\therefore Y(z) = \frac{4z^2}{z^2+z-2} + \frac{2z^2-z}{z^2+z-2} X(z)$$

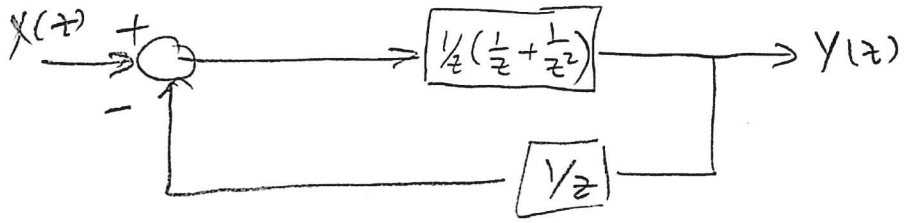
4 Find the transfer function of the following system

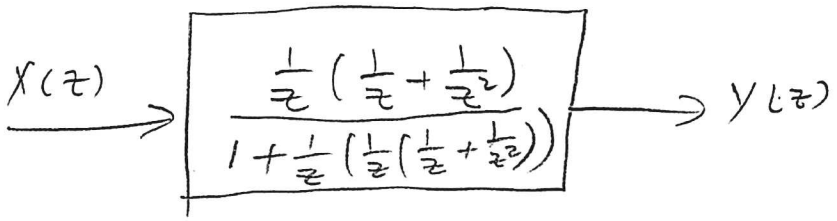












$$\therefore H(z) = \frac{\frac{1}{z}\left(\frac{1}{z}+\frac{1}{z^2}\right)}{1+\frac{1}{z}\left(\frac{1}{2}\left(\frac{1}{z}+\frac{1}{z^2}\right)\right)}$$

$$H(z) = \frac{\frac{1}{z^2}+\frac{1}{z^3}}{1+\frac{1}{z^3}+\frac{1}{z^4}} \times \frac{z^4}{z^4}$$

$$\therefore H(z) = \frac{z^2+z}{z^4+z+1}$$

OR $$y(z) = X_1(z)\left[\frac{1}{z}\left(\frac{1}{z}+\left(\frac{1}{z}\times\frac{1}{z}\right)\right)\right]$$

$$= X_1(z)\left[\frac{1}{z^2}+\frac{1}{z^3}\right]$$

$$X_1(z) = X(z) - \frac{1}{z}Y(z)$$

$$\therefore Y(z) = \left(X(z) - \frac{1}{z}Y(z)\right)\left(\frac{1}{z^2}+\frac{1}{z^3}\right)$$

$$Y(z) = X(z)\left(\frac{1}{z^2} + \frac{1}{z^3}\right) - Y(z)\left(\frac{1}{z}\right)\left(\frac{1}{z^2} + \frac{1}{z^3}\right)$$

$$= X(z)\left(\frac{1}{z^2} + \frac{1}{z^3}\right) - Y(z)\left(\frac{1}{z^3} + \frac{1}{z^4}\right)$$

$$\therefore Y(z)\left(1 + \frac{1}{z^3} + \frac{1}{z^4}\right) = X(z)\left(\frac{1}{z^2} + \frac{1}{z^3}\right)$$

$$\therefore \frac{Y(z)}{X(z)} = H(z) = \frac{\frac{1}{z^2} + \frac{1}{z^3}}{1 + \frac{1}{z^3} + \frac{1}{z^4}} \quad * \quad \frac{z^4}{z^4}$$

$$\therefore H(z) = \frac{z^2 + z}{z^4 + z + 1} \quad *$$

(5) Is the following stable?

$$y[n+2] - y[n+1] + y[n] = x[n+2] - x[n+1]$$

$$z^2 Y(z) - z Y(z) + Y(z) = z^2 X(z) - z X(z)$$

$$\therefore (z^2 - z + 1) Y(z) = (z^2 - z) X(z)$$

$$\therefore H(z) = \frac{z^2 - z}{z^2 - z + 1}$$

The poles are $0.5 \pm j0.866$

The magnitude of the poles is 1.0

$\Rightarrow$ marginally stable

(6) Is the following system stable?

$$H(z) = \frac{z-4}{z^2+1.5z+0.5} = \frac{z-4}{(z+1)(z+0.5)}$$

$\Rightarrow$ poles $-1 \& -0.5$

Non-repeated real pole on the unit circle $\Rightarrow$ marginally stable

(7) Is the following system stable?

$$H(z) = \frac{z-3}{z^2+3z+2}$$

$$\therefore H(z) = \frac{z-3}{(z+1)(z+2)}$$

$\Rightarrow$ poles $-1 \& -2$ $\Rightarrow$ outside the unit disk $\Rightarrow$ unstable

(8) Is the following system stable?

$$H(z) = \frac{z+2}{z^2+z+0.16}$$

$$\therefore H(z) = \frac{z+2}{(z+0.8)(z+0.2)}$$

$\Rightarrow$ poles $-0.8 \& -0.2$

$\therefore$ Poles inside the unit disk $\Rightarrow$ stable

EE 362 Review session #1

z-Transform

(1) For the following LTI DT system,

$$y[n] + 1.5y[n-1] + 0.56y[n-2] = x[n] + 2x[n-1]$$

do the following

(a) Find the transfer function of the system

(b) Draw the single line flow diagram

(c) Find the transfer function of part (b)

(d) compare the transfer functions of parts (a) and (c)

"Solution"

(a) Since we want to find the transfer function, all IC's are zero

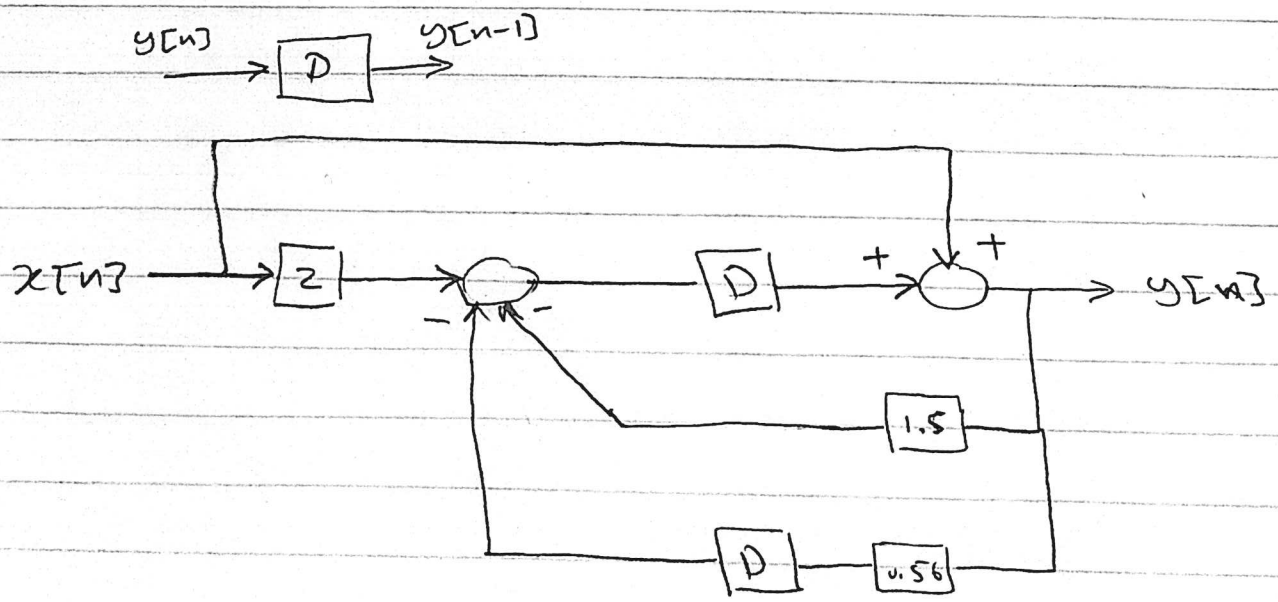
$$\therefore Y(z) + 1.5z^{-1}Y(z) + 0.56z^{-2}Y(z) = X(z) + 2z^{-1}X(z)$$

$$Y(z)[1 + 1.5z^{-1} + 0.56z^{-2}] = X(z)[1 + 2z^{-1}]$$

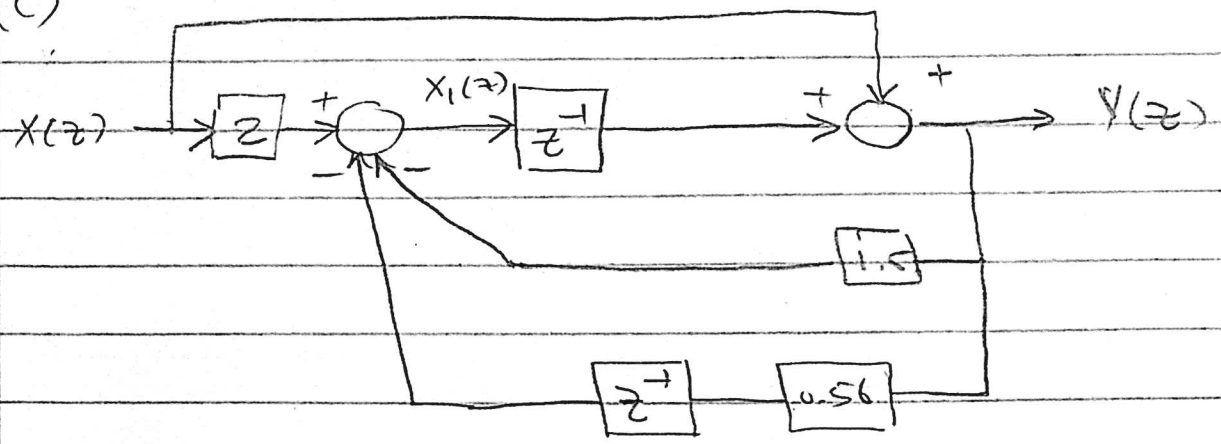
$$\therefore H(z) = \frac{1 + 2z^{-1}}{1 + 1.5z^{-1} + 0.56z^{-2}} \times \frac{z^2}{z^2}$$

$$\therefore H(z) = \frac{z^2 + 2z}{z^2 + 1.5z + 0.56}$$

(b) $y[n] = -1.5y[n-1] - 0.56y[n-2] + x[n] + 2x[n-1]$



(c)



$$Y(z) = z^{-1} X_1(z) + X(z)$$

$$X_1(z) = 2X(z) - 1.5\,Y(z) - 0.56\,z^{-1} Y(z)$$

$$\therefore Y(z) = z^{-1}\left(2X(z) - 1.5\,Y(z) - 0.56\,z^{-1}Y(z)\right) + X(z)$$

$$\therefore Y(z) = 2z^{-1}X(z) - 1.5\,z^{-1}Y(z) - 0.56\,z^{-2}Y(z) + X(z) \times z^2$$

$$\therefore z^2 Y(z) = 2z\,X(z) - 1.5\,z\,Y(z) - 0.56\,Y(z) + z^2 X(z)$$

$$(z^2 + 1.5z + 0.56)\,Y(z) = (z^2 + 2z)\,X(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{z^2 + 2z}{z^2 + 1.5z + 0.56}$$

(d) They are identical

(e) Is the system of problem (1) stable?

$$H(z) = \frac{z^2 + 2z}{(z + 0.7)(z + 0.8)}$$

Since the magnitudes of all poles are smaller than 1.0 then, the system is stable.

EE 362

3

Z - Transform

(1) A discrete-time signal $x[n]$ has z-Transform

$$X(z) = \frac{z+1}{z(z-1)}$$

Compute $x[0]$, $x[1]$, and $x[10000]$.

$$X(z) = \frac{c_1}{z} + \frac{c_2}{z-1}$$

$$c_1 = z\,X(z)\Big|_{z=0} = \frac{1}{-1} = -1$$

$$c_2 = (z-1)\,X(z)\Big|_{z=+1} = 2$$

$$\therefore X(z) = -\frac{1}{z} + \frac{2}{z-1} = -z^{-1}(1) + 2z^{-1}\left(\frac{z}{z-1}\right)$$

$$\therefore x[n] = -\delta[n-1] + 2u[n-1]$$

$$\Rightarrow x[0] = -\delta[-1] + 2u[-1] = 0$$

$$x[1] = -\delta[0] + 2u[0] = 1$$

$$x[10000] = -\delta[9999] + 2u[9999] = 0 + 2 = 2$$

(2) Compute the inverse z-transform of the following signals

(a) $\frac{z}{z^2+1}$ $\Rightarrow$ using table 7.3

$$\sin(\Omega n)\,u[n] \longleftrightarrow \frac{\sin(\Omega)\,z}{z^2 - (2\cos\Omega)\,z + 1}$$

if $\Omega = \frac{\pi}{2} \Rightarrow \sin(\Omega) = 1$ & $\cos(\Omega) = 0$

4

$\therefore x[n] = \sin\left(\frac{\pi}{2}n\right), \; n \geq 0$

(b) $X(z) = \frac{z^2}{z^2+1} = z\left(\frac{z}{z^2+1}\right)$

$\sin\left(\frac{\pi}{2}n\right) \Longleftrightarrow \frac{z}{z^2+1}$

By 6f shift property $\Rightarrow x[n] = \sin\left(\frac{\pi}{2}(n+1)\right), \; n \geq 0$

$x[n] = \sin\left(\frac{\pi}{2}n + \frac{\pi}{2}\right), \; n \geq 0$

$= \cos\left(\frac{\pi}{2}n\right), \; n \geq 0$

(c) $X(z) = \frac{1}{z^2+1} + \frac{1}{z^2-1} = \frac{1}{z^2+1} + \frac{1}{(z-1)(z+1)}$

$= z^{-1}\left(\frac{z}{z^2+1}\right) + \frac{C_1}{(z-1)} + \frac{C_2}{(z+1)}$

$C_1 = (z-1)\left(\frac{1}{(z-1)(z+1)}\right)\Big|_{z=1} = \frac{1}{2}$

$C_2 = (z+1)\left(\frac{1}{(z-1)(z+1)}\right)\Big|_{z=-1} = -\frac{1}{2}$

$\therefore X(z) = z^{-1}\left(\frac{z}{z^2+1}\right) + \frac{1}{2}\frac{z^{-1}(z)}{z-1} - \frac{1}{2}\frac{z^{-1}(z)}{z+1}$

$\therefore x[n] = \sin\left(\frac{\pi}{2}(n-1)\right)u[n-1] + \frac{1}{2}(1)^{n-1}u[n-1] - \frac{1}{2}(-1)^{n-1}u[n-1]$

...

(d) $X(z) = \frac{z^2}{z^2+1} + \frac{z}{z^2-1}$

5

$$\frac{z}{z^2-1} = z\left(\frac{1}{z^2-1}\right) = z\left(\frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z+1}\right)$$

$$\therefore \frac{1}{z^2-1} \leftrightarrow \frac{1}{2}\left[1-(-1)^{n-1}\right]u[n-1]$$

$$\Rightarrow z\left(\frac{1}{z^2-1}\right) \leftrightarrow \frac{1}{2}\left(1-(-1)^{n}\right)u[n]$$

From table 7.3, $\cos(\Omega n)\,u[n] \leftrightarrow \dfrac{z^2-\cos(\Omega)z}{z^2-(2\cos(\Omega))z+1}$

If $\Omega = \frac{\pi}{2} \Rightarrow \cos(\Omega) = 0$

$$\therefore \frac{z^2}{z^2+1} \leftrightarrow \cos\left(\frac{\pi}{2}n\right)$$

$$x[n] = \cos\left(\frac{\pi}{2}n\right) + \frac{1}{2}\left(1-(-1)^n\right), \quad n \geq 0$$

(e) $$X(z) = \frac{z^2+1}{z^2+1} = \frac{z^2}{z^2+1} - \frac{1}{z^2+1}$$

$$= \frac{z^2}{z^2+1} - z^{-1}\left(\frac{z}{z^2+1}\right)$$

$$\frac{z}{z^2+1} \leftrightarrow \sin\left(\frac{\pi}{2}n\right)u[n]$$

$$\therefore z^{-1}\left(\frac{z}{z^2+1}\right) \leftrightarrow \sin\left(\frac{\pi}{2}(n-1)\right)u[n-1]$$

$$\leftrightarrow \sin\left(\frac{\pi}{2}n - \frac{\pi}{2}\right) = -\cos\left(\frac{\pi}{2}n\right)u[n-1]$$

6

$$\therefore \frac{1}{z^2+1} \longleftrightarrow -\cos\left(\frac{\pi}{2}n\right)u[n-1]$$

$$\therefore x[n] = \cos\left(\frac{\pi}{2}n\right)u[n] + \cos\left(\frac{\pi}{2}n\right)u[n-1]$$

$$= \cos(0)\,u[0] + \cos\left(\frac{\pi}{2}n\right)u[n-1] + \cos\left(\frac{\pi}{2}n\right)u[n-1]$$

$$= \delta[n] + 2\cos\left(\frac{\pi}{2}n\right)u[n-1]$$

(f) $$X(z) = \frac{z+2}{(z-1)(z^2+1)}$$

$$\frac{X(z)}{z} = \frac{C_1}{z} + \frac{C_2}{(z-1)} + \frac{C_3}{z-j} + \frac{C_4}{z+j}$$

$$C_1 = z\left(\frac{X(z)}{z}\right)\Big|_{z=0} = -2$$

$$C_2 = (z-1)\left(\frac{X(z)}{z}\right)\Big|_{z=1} = \frac{3}{2}$$

$$C_3 = (z-j)\left(\frac{X(z)}{z}\right)\Big|_{z=j} = \frac{j+2}{j(j-1)(j+j)}$$

$$= \frac{j+2}{-2j+2} = \frac{(j+2)(2j+2)}{4+4} = \frac{2+6j}{8} = \frac{1}{4}(1+j3)$$

$$\therefore \frac{X(z)}{z} = \frac{-2}{z} + \frac{3/2}{z-1} + \frac{1/4(1+j3)}{z-j} + \frac{1/4(1-j3)}{z+j}$$

For complex poles $2|C_1|\sigma^n\cos(\Omega n + \angle C_1)$

$\sigma = |P_1| = |j| = 1$, $\Omega = \angle P_1 = \frac{\pi}{2}$

$$|C_1| = \frac{1}{4}\sqrt{1+9} = \frac{\sqrt{10}}{4}, \quad \angle C_1 = \tan^{-1}\left(\frac{3}{1}\right) = 1.249 \text{ rad}$$

$$X(z) = -2 + \frac{3}{2}\left(\frac{z}{z-1}\right) + \frac{1}{4}(1+j3)\frac{z}{z-j} + \frac{1}{4}(1-j3)\frac{z}{z+j}$$

$$\therefore x[n] = -2\delta[n] + \frac{3}{2} + \frac{\sqrt{10}}{2}\cos\left(\frac{\pi}{2}n + 1.249\right), \quad n \geq 0$$

(g) $$X(z) = \frac{z^2+2}{(z-1)\left(\frac{z}{z}+1\right)}$$

$$\therefore X(z) = \frac{-2}{z} + \frac{3/2}{z-1} + \frac{1/4(1+j)}{z-j} + \frac{1/4(1-j)}{z+j}$$

$$\sigma = 1, \quad \Omega = \frac{\pi}{2}, \quad |C_1| = \frac{\sqrt{2}}{4}, \quad \angle C_1 = \frac{\pi}{4}$$

$$\therefore x[n] = -2\delta[n] + \frac{3}{2} + \frac{\sqrt{2}}{2}\cos\left(\frac{\pi}{2}n + \frac{\pi}{4}\right), \quad n \geq 0$$

(h) $$nx[n] \leftrightarrow -z\frac{d}{dz}\left[\ln\left(\frac{2z-1}{2z}\right)\right]$$

$$\leftrightarrow -z\left(\frac{2z}{2z-1}\right)\left(\frac{1}{2z^2}\right)$$

$$\leftrightarrow \frac{-0.5}{z-0.5} = -0.5z^{-1}\left(\frac{z}{z-0.5}\right)$$

$$\therefore n\,x[n] = -0.5(0.5)^{n-1}u[n-1] = -(0.5)^n u[n-1]$$

$$\therefore x[n] = \frac{-(0.5)^n}{n}u[n-1]$$

(3) Compute the convolution $x[n] * v[n]$ for all $n \geq 0$ using z-transform

(a) $x[n] = u[n] + 3\delta[n-1]$, $v[n] = u[n-2]$

$$X(z) = \frac{z}{z-1} + 3z^{-1} \quad \& \quad V(z) = z^{-2}\left(\frac{z}{z-1}\right) = \frac{z^{-1}}{z-1}$$

$$X(z)V(z) = \frac{1}{(z-1)^2} + \frac{3z^{-2}}{z-1} \leftrightarrow (n-1)u[n-1] + 3u[n-3]$$

(b) $x[n] = u[n]$ & $v[n] = n\,u[n]$

$X(z) = \frac{z}{z-1}$ & $V(z) = \frac{z}{(z-1)^2}$

$X(z)\,V(z) = \frac{z^2}{(z-1)^3} \Leftrightarrow \frac{1}{2}\,n(n+1)\,u[n]$

(4) Determine whether or not the following LTI DT systems are stable

(a) $\frac{1}{z^3 + 0.5z + 0.1}$

roots of $A(z) = 0$ are $-0.0935 + j\,0.7254$

$-0.0935 - j\,0.7254$

-0.1869

The magnitudes of the poles are:

$0.7314,\ 0.7314$ & $0.1869 < 1$

$\Rightarrow$ stable

(5) Is this LTI DT system stable?

$y[n+2] - y[n+1] + y[n] = x[n+2] - x[n+1]$

$z^2 Y(z) - z\,Y(z) + Y(z) = z^2 X(z) - z\,X(z)$

$(z^2 - z + 1)\,Y(z) = (z^2 - z)\,X(z)$

$H(z) = \frac{z^2 - z}{z^2 - z + 1} \Rightarrow$ poles are: $0.5 \pm j\,0.866$

Magnitude $= 1 \Rightarrow$ marginally stable

Examples:

① $x[n] = \delta[n]$

$$X(z) = \sum_{n=0}^{\infty} \delta[n] z^{-n} = \delta[0] z^{-0} = 1 \quad \Rightarrow \boxed{\delta[n] \longleftrightarrow 1}$$

②

$x[n] = \delta[n-q]$, where q is positive integer

$$X(z) = \sum_{n=0}^{\infty} \delta[n-q] z^{-n} = \delta[0] z^{-q} = \frac{1}{z^q} \quad \Rightarrow \boxed{\delta[n-q] \longleftrightarrow \frac{1}{z^q}}$$

③

$x[n] = u[n]$

$$X(z) = \sum_{n=0}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = 1 + z^{-1} + z^{-2} + \cdots$$

$$\therefore (z-1) X(z) = (z-1)(1 + z^{-1} + z^{-2} + \cdots)$$

$$= (z + 1 + z^{-1} + z^{-2} \cdots) - (1 + z^{-1} + z^{-2} + \cdots)$$

$$= z$$

$$\Rightarrow X(z) = \frac{z}{z-1} \quad \Rightarrow \boxed{u[n] \longleftrightarrow \frac{z}{z-1}}$$

④

$x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = 1 + a z^{-1} + a^2 z^{-2} + \cdots$$

$$\therefore (z-a) X(z) = (z-a)(1 + a z^{-1} + a^2 z^{-2} + \cdots)$$

$$= (z + a + a^2 z^{-1} + \cdots) - (a + a^2 z^{-1} + a^3 z^{-2} + \cdots)$$

$$= z$$

$$\therefore X(z) = \frac{z}{z-a} \quad \Rightarrow \boxed{a^n u[n] = \frac{z}{z-a}}$$

⑤ $x[n] = u[n] + a^n u[n]$

$$X(z) = \frac{z}{z-1} + \frac{z}{z-a} = \frac{2z^2 - (1+a)z}{(z-1)(z-a)}$$

⑥ $x[n] = u[n] - u[n-2]$

$$X(z) = \frac{z}{z-1} - \frac{z}{z-1} z^{-2} = \frac{z - z^{-1}}{z-1}$$

⑦ $x[n] = u[n+1]$

$$X(z) = z X(z) - x[0] z$$

$$= z\left(\frac{z}{z-1}\right) - z = \frac{z^2 - z^2 + z}{z-1} = \frac{z}{z-1}$$

⑧ $x[n] = n a^n u[n]$

Let $v[n] = a^n u[n] \Rightarrow V(z) = \frac{z}{z-a}$

$$\therefore X(z) = -z \frac{d}{dz} V(z) = -z \frac{d}{dz}\left(\frac{z}{z-a}\right)$$

$$= -z \frac{(z-a)(1) - (z)(1)}{(z-a)^2} = -\frac{z(-a)}{(z-a)^2}$$

$$\therefore X(z) = \frac{az}{(z-a)^2}$$

If $a=1 \Rightarrow X(z) = \frac{z}{(z-a)^2} \Leftrightarrow nu[n]$

$$\Rightarrow \boxed{na^n u[n] \Leftrightarrow \frac{az}{(z-a)^2}} \neq \boxed{nu[n] \Leftrightarrow \frac{z}{(z-a)^2}}$$

(9)

$x[n] = n^2 a^n u[n]$

let $v[n] = n a^n u[n] \Rightarrow V(z) = \frac{az}{(z-a)^2}$ (from Example 8)

$\therefore X(z) = -z \frac{d}{dz} V(z) = -z \frac{d}{dz}\left(\frac{az}{(z-a)^2}\right)$

$= \frac{-z[(z-a)^2(a) - (az)(2(z-a))]}{(z-a)^4}$

$= \frac{-z(z-a)[az - a^2 - 2az]}{(z-a)^4} = -z\,\frac{-az-a^2}{(z-a)^3}$

$= \frac{az^2 + a^2 z}{(z-a)^3} = \frac{az(z+a)}{(z-a)^3}$

If $a=1$, $X(z) = \frac{z(z+1)}{(z-1)^3}$

$\therefore$ $n^2 a^n u[n] \leftrightarrow \frac{az(z+a)}{(z-a)^3}$ $\neq$ $n^2 u[n] \leftrightarrow \frac{z(z+1)}{(z-1)^3}$

(10) $x[n] = a^n p[n]$ where $p[n] = u[n] - u[n-q]$

$x[n] = a^n (u[n] - u[n-q])$

$u[n] - u[n-q] \leftrightarrow \frac{z(1-z^{-q})}{z-1}$

$\therefore X(z) = \frac{(z/a)(1-(z/a)^{-q})}{z/a - 1} = \frac{z(1-a^q z^{-q})}{z-a}$

or $a^n u[n] \Leftrightarrow \frac{z}{z-a}$

$u[n-q] \Leftrightarrow \frac{z}{z-1} z^{-q} = \frac{z^{-q+1}}{z-1}$

$a^n u[n-q] \Leftrightarrow \frac{(z/a)^{-q+1}}{(z/a)-1} = \frac{z^{-q+1} a^{q-1}}{z/a-1} = \frac{a^q z^{-q+1}}{z-a}$

$\Leftrightarrow \frac{z a^q z^{-q}}{z-a}$

$\therefore X(z) = \frac{z}{z-a} - \frac{z a^q z^{-q}}{z-a} = \frac{z(1-a^q z^{-q})}{z-a}$ ✱

III $x[n] = \cos(\Omega n) u[n]$

Let $v[n] = u[n] \Rightarrow V(z) = \frac{z}{z-1}$

$\therefore X(z) = \frac{1}{2}\left(\frac{e^{j\Omega} z}{e^{j\Omega} z - 1} + \frac{e^{-j\Omega} z}{e^{-j\Omega} z - 1}\right)$

$= \frac{1}{2}\left[\frac{e^{j\Omega} z(e^{-j\Omega} z - 1) + e^{-j\Omega} z(e^{j\Omega} z - 1)}{(e^{j\Omega} z - 1)(e^{-j\Omega} z - 1)}\right]$

$= \frac{1}{2}\left[\frac{z^2 - e^{j\Omega} z + z^2 - e^{-j\Omega} z}{z^2 - z(e^{j\Omega} + e^{-j\Omega}) + 1}\right]$

$= \frac{1}{2}\left[\frac{2z^2 - 2\cos(\Omega) z}{z^2 - 2\cos(\Omega) z + 1}\right] = \frac{z^2 - \cos(\Omega) z}{z^2 - 2\cos(\Omega) + 1}$

[12] $x[n] = a^n \cos(\Omega n)\, u[n]$

Let $V[n] = \cos(\Omega n)\, u[n]$

$$\Rightarrow V(z) = \frac{z^2 - \cos(\Omega) z}{z^2 - 2\cos(\Omega) z + 1}$$

$$\therefore X(z) = \frac{(z/a)^2 - \cos(\Omega)(z/a)}{(z/a)^2 - 2\cos(\Omega)(z/a) + 1}$$

$$= \frac{z^2 - a\cos(\Omega) z}{z^2 - 2a\cos(\Omega) z + a^2}$$

[13] $$X(z) = \frac{z}{8z^2 - 2z + 1}$$

$V[n] = e^{3n} x[n]$

$V(z) = ?$

$$V(z) = X(e^{-3} z) = \frac{e^{-3} z}{8e^{-6} z^2 - 2e^{-3} z + 1}$$

[14] $x[n] = e^{0.5n} u[n] + u[n-2]$

$$X(z) = \frac{z}{z - e^{0.5}} + \frac{z}{z-1} z^{-2}$$

14

(1) A system is described by the difference equation

$$y[n] + 0.5\, y[n-1] = u[n]; \quad y[-1] = 0$$

Find an analytical expression for $y[n]$ and the transfer function of the system

$$Y(z) + 0.5\, z^{-1} Y(z) = \frac{z}{z-1}$$

$$Y(z) = \left(\frac{z}{z-1}\right)\left(\frac{1}{1+0.5z^{-1}}\right) = \left(\frac{z}{z-1}\right)\underbrace{\left(\frac{z}{z+0.5}\right)}_{H(z)}$$

$$= \frac{z^2}{(z-1)(z+0.5)}$$

$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z+0.5)} = \frac{C_1}{z-1} + \frac{C_2}{z+0.5}$$

$$C_1 = (z-1)\left.\frac{Y(z)}{z}\right|_{z=1} = \left.\frac{z}{z+0.5}\right|_{z=1} = \frac{2}{3}$$

$$C_2 = (z+0.5)\left.\frac{Y(z)}{z}\right|_{z=-0.5} = \left.\frac{z}{z-1}\right|_{z=-0.5} = \frac{1}{3}$$

$$\therefore Y(z) = \frac{2}{3}\left(\frac{z}{z-1}\right) + \frac{1}{3}\left(\frac{z}{z+0.5}\right)$$

$$\therefore y[n] = \frac{2}{3}u[n] + \frac{1}{3}(-0.5)^n u[n]$$

or $y[n] = \frac{2}{3} + \frac{1}{3}(-0.5)^n, \quad n \geq 0$

$$\boxed{H(z) = \frac{z}{z+0.5}}$$

(2) Find the transfer function of the following system if all ICs = 0

$$y[n] - 0.2\,y[n-1] - 0.8\,y[n-2] = 2x[n] + x[n-1]$$

$$Y(z) - 0.2\,z^{-1}Y(z) - 0.8\,z^{-2}Y(z) = 2X(z) + z^{-1}X(z)$$

$$\therefore Y(z)\left[1 - 0.2z^{-1} - 0.8z^{-2}\right] = X(z)\left[2 + z^{-1}\right]$$

$$\therefore \frac{Y(z)}{X(z)} = H(z) = \frac{2 + z^{-1}}{1 - 0.2z^{-1} - 0.8z^{-2}} \times \frac{z^2}{z^2}$$

$$\therefore H(z) = \frac{2z^2 + z}{z^2 - 0.2z - 0.8}$$

(3) A LTI DT system is given by the input/output difference equation

$$y[n] + y[n-1] - 2y[n-2] = 2x[n] - x[n-1] \quad -(1)$$

Find $Y(z)$ if $y[-2] = 2$ and $y[-1] = 0$

- From (1):

$$Y(z) + z^{-1}Y(z) + y[-1] - 2\left(z^{-2}Y(z) + y[-2] + z^{-1}y[-1]\right)$$
$$= 2X(z) - z^{-1}X(z) - x[-1]$$

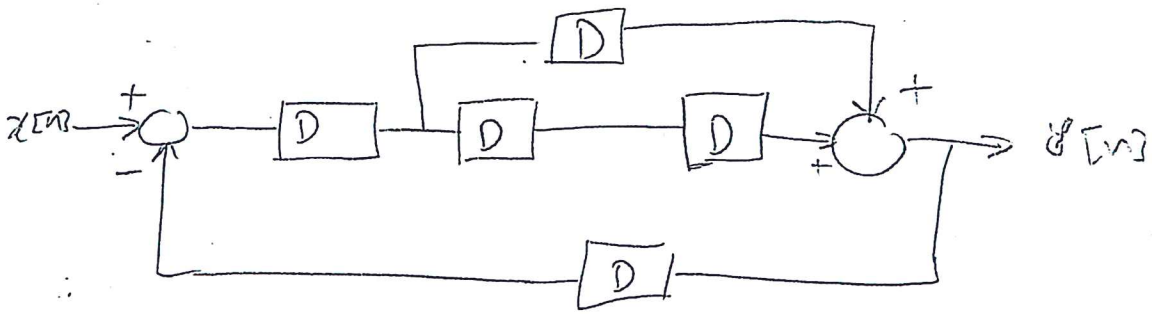
$$\therefore Y(z) + z^{-1}Y(z) - 2z^{-2}Y(z) - 4 = 2X(z) - z^{-1}X(z)$$

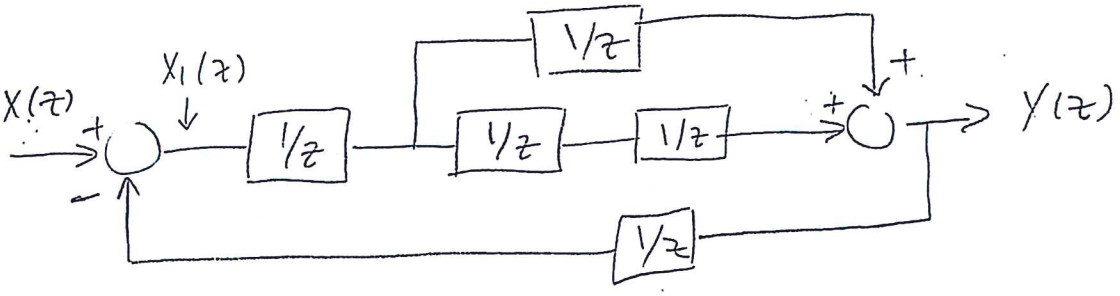
$$Y(z)\left(1 + z^{-1} - 2z^{-2}\right) - 4 = X(z)\left(2 - z^{-1}\right)$$

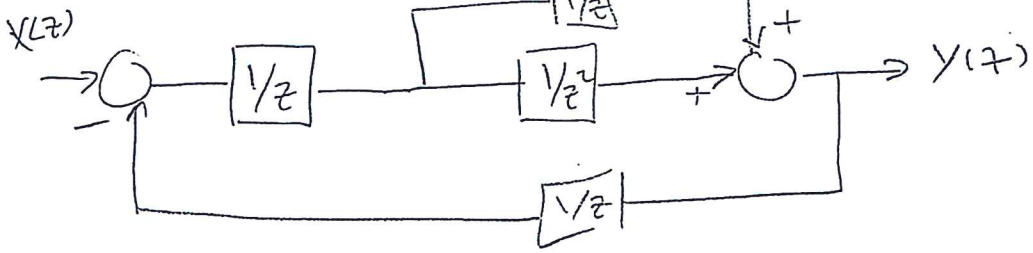
$$\therefore Y(z) = \frac{4}{1 + z^{-1} - 2z^{-2}} + \frac{2 - z^{-1}}{1 + z^{-1} - 2z^{-2}}X(z)$$

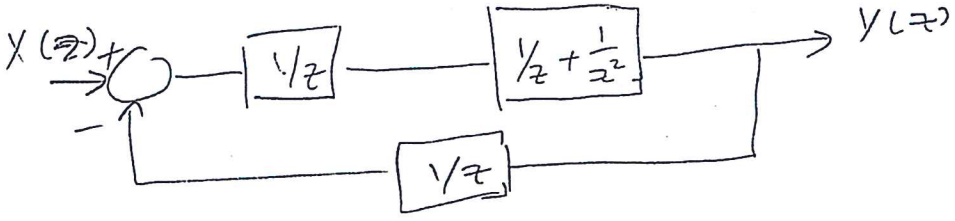
$$\therefore Y(z) = \frac{4z^2}{z^2+z-2} + \frac{2z^2-z}{z^2+z-2} X(z)$$

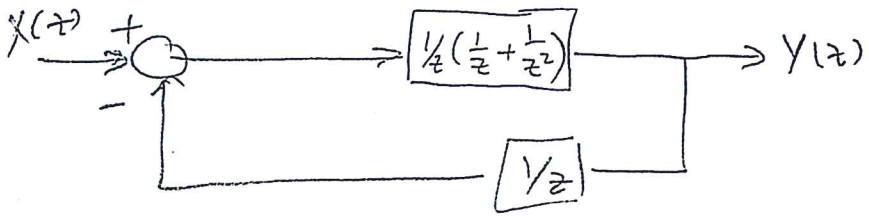
4. Find the transfer function of the following system

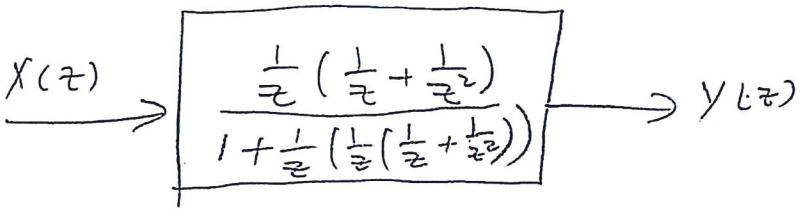












$$\therefore H(z) = \frac{\frac{1}{z}\left(\frac{1}{z}+\frac{1}{z^2}\right)}{1+\frac{1}{z}\left(\frac{1}{z}\left(\frac{1}{z}+\frac{1}{z^2}\right)\right)}$$

$$H(z) = \frac{\frac{1}{z^2}+\frac{1}{z^3}}{1+\frac{1}{z^3}+\frac{1}{z^4}} \times \frac{z^4}{z^4}$$

$$\therefore H(z) = \frac{z^2+z}{z^4+z+1}$$

OR $$y(z) = X_1(z)\left[\frac{1}{z}\left(\frac{1}{z}+\left(\frac{1}{z}\times\frac{1}{z}\right)\right)\right]$$

$$= X_1(z)\left[\frac{1}{z^2}+\frac{1}{z^3}\right]$$

$$X_1(z) = X(z) - \frac{1}{z}Y(z)$$

$$\therefore Y(z) = \left(X(z) - \frac{1}{z}Y(z)\right)\left(\frac{1}{z^2}+\frac{1}{z^3}\right)$$

$$Y(z) = X(z)\left(\frac{1}{z^2}+\frac{1}{z^3}\right) - Y(z)\left(\frac{1}{z}\right)\left(\frac{1}{z^2}+\frac{1}{z^3}\right)$$

$$= X(z)\left(\frac{1}{z^2}+\frac{1}{z^3}\right) - Y(z)\left(\frac{1}{z^3}+\frac{1}{z^4}\right)$$

$$\therefore Y(z)\left(1+\frac{1}{z^3}+\frac{1}{z^4}\right) = Y(z)\left(\frac{1}{z^2}+\frac{1}{z^3}\right)$$

$$\therefore \frac{Y(z)}{X(z)} = H(z) = \frac{\frac{1}{z^2}+\frac{1}{z^3}}{1+\frac{1}{z^3}+\frac{1}{z^4}} \quad * \frac{z^4}{z^4}$$

$$\therefore H(z) = \frac{z^2+z}{z^4+z+1} \quad \ast$$

(5) Is the following stable?

$$y[n+2] - y[n+1] + y[n] = x[n+2] - x[n+1]$$

$$z^2Y(z) - zY(z) + Y(z) = z^2X(z) - zX(z)$$

$$\therefore (z^2-z+1)\,Y(z) = (z^2-z)\,X(z)$$

$$\therefore H(z) = \frac{z^2-z}{z^2-z+1}$$

The poles are $0.5 \pm j0.866$

The magnitude of the poles is 1.0

$\Rightarrow$ marginally stable

(6) Is the following system stable?

$$H(z) = \frac{z-4}{z^2+1.5z+0.5} = \frac{z-4}{(z+1)(z+0.5)}$$

$\Rightarrow$ poles $-1 \& -0.5$

Non-repeated real pole on the unit circle $\Rightarrow$ marginally stable

(7) Is the following system stable?

$$H(z) = \frac{z-3}{z^2+3z+2}$$

$$\therefore H(z) = \frac{z-3}{(z+1)(z+2)}$$

$\Rightarrow$ poles $-1 \& -2$ $\Rightarrow$ outside the unit disk $\Rightarrow$ unstable

(8) Is the following system stable?

$$H(z) = \frac{z+2}{z^2+z+0.16}$$

$$\therefore H(z) = \frac{z+2}{(z+0.8)(z+0.2)}$$

$\Rightarrow$ poles $-0.8 \& -0.2$

$\therefore$ poles inside the unit disk $\Rightarrow$ stable